PREPARATION DE RENTREE — Session 2020

Exercices préparatoires Performance Engineering

Physical Quantities & Units

Definitions of Physical Quantities

The correct usage of scientific / engineering terms is critically important for communication and understanding. You should be able to define each of the following physical quantities in a simple but complete manner (in French, at least, but ideally in English as well) along with their SI units.

Distance & Displacement

Mass

Temperature

Force

Torque, Moment and Couple

Pressure

Work

Energy

Power

Scalar & Vector Quantities

Physical quantities fall into one of two categories: scalar quantities and vector quantities. Scalar quantities have a value of size, or magnitude. Vector quantities have both magnitude and direction.

Specify which of the following quantities are scalar and which are vectors:

Quantity	**Scalar / Vector**
Mass	
Velocity	
Volume	
Displacement	
Acceleration	
Force	
Pressure	
Distance	

Moment & Torque

Moment and torque are quantities that have magnitude and direction. However, since their direction is defined by *convention* rather than physics, they are not, strictly, true vector quantities but *psuedovectors*. The difference is only practically important when dealing with reflections or inversions in a coordinate system. More importantly, moment and torque should always be described by a magnitude and direction.

SI Units

SI Base Units

These seven quantities and their units are considered the fundamental, independent dimensions. All other SI units are derived from these base units. Complete the table below; the first line has been completed as an example:

Physical Quantity	**SI Unit**	**Symbol**
Length	metre*	m

* n.b. UK English spelling is *metre*; US English is *meter*.

SI Derived Units

The rest of the SI units are either dimensionless or derived from one or more of the base units. Complete the table of common and relevant SI derived units below:

Physical Quantity	SI Derived Unit	Symbol
Area		
Volume		
speed, velocity		
Acceleration		
angular velocity		
angular acceleration		
Momentum		
angular momentum		
torque, moment of force		
	hertz	
	radian	
	newton	
Pressure		
energy, work, heat		
Power		
voltage, electrical potential		
		Ω
		kg/m^3
heat capacity		
dynamic viscosity		
thermal conductivity		
moment of inertia		

SI Prefixes

The SI prefixes represent multiples / submultiples of the SI units, allowing large / small values to be written more conveniently. There are 20 SI prefixes representing multiples of 10^{-24} to 10^{24}.

Complete the following table of selected SI prefixes. The first row is filled as an example:

Prefix	Name	Significance
T	tera	multiply by 10^{12}
		multiply by 10^{9}
		multiply by 10^{6}
		multiply by 10^{3}
		multiply by 10^{2}
		multiply by 10^{1}
		multiply by 10^{-1}
		multiply by 10^{-2}
		multiply by 10^{-3}
		multiply by 10^{-6}

Note the special case of kilogram, which includes a prefix in the SI unit itself. In this case, the prefixes are applied to the unit of gram.

Work, energy, power

You have already reviewed the definitions of these physical quantities. It is important to know how they relate to each other. You should know how the relationships between the following physical quantities:

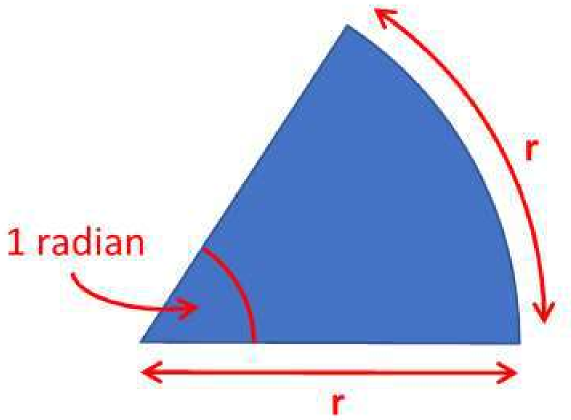
Work and force	
Work and torque	
Power and work	
Power and force	
Power and torque	
Power and velocity	
Power and angular velocity	

Trigonometry

Radians and Degrees

The *radian* is the standard mathematical unit for the measurement of an angle. It is represented by the symbol **rad** (or the angle is written without a unit).

One radian is the angle created between the ends of an *arc* of length equal to the radius of a circle:



Therefore, an angle in radians is equal to the ratio of the arc length to the radius:

$$\theta = s/r$$

Where:

- θ = angle in radians
- s = arc length
- r = radius

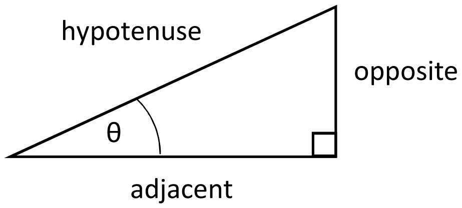
For one full rotation, the arc length is equal to the circumference of the circle. Since the circumference is equal to 2πr, a full rotation measured in radians is:

$$\theta = 2\pi r/r = 2\pi$$

The *degree* is another unit of angular measurement. You should be comfortable working in both degrees and radians, converting between one and the other, and selecting the most applicable unit for the task at hand.

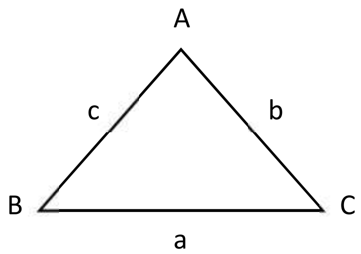
Right-angled triangles

You should be comfortable with the trigonometric functions (sine, cosine and tangent ratios) and Pythagoras' theorem. English right-angled triangle nomenclature is similar to French:



Sine and Cosine Rules

For any triangle:



The sine rule is:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The cosine rule is:

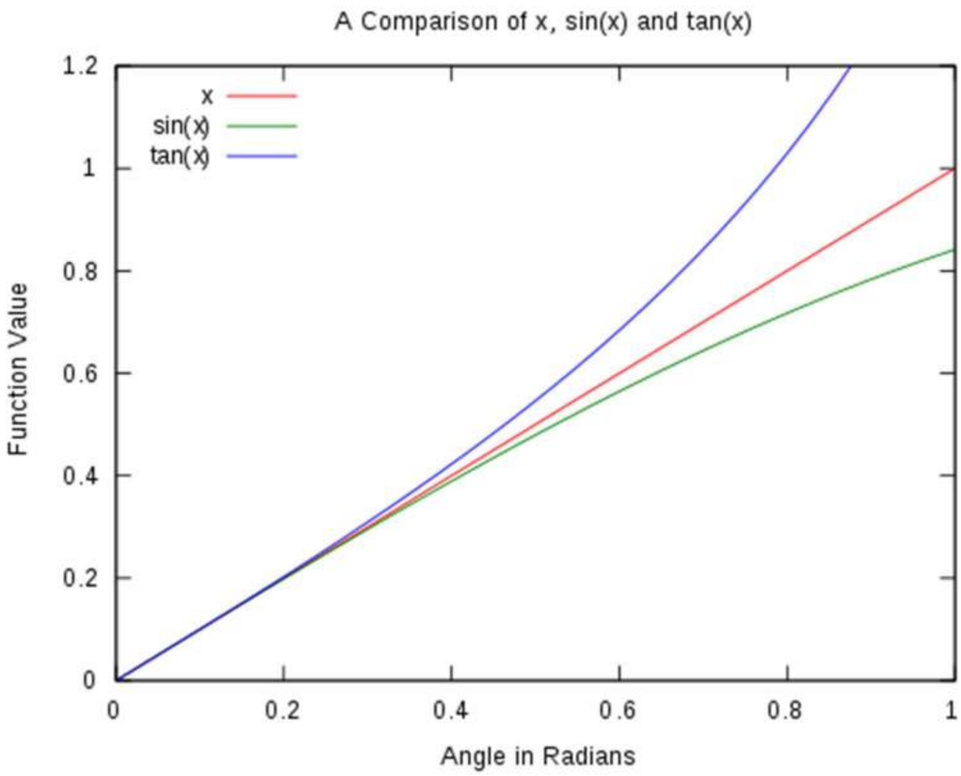
$$a^2 = b^2 + c^2 - 2bc\cos A$$

Small angle approximation

The small angle approximation utilises the fact that the trigonometric functions sine, cosine and tangent each closely approach a simpler function as the angle approaches zero.

Sin ϑ and Tan ϑ

As the angle θ approaches zero, the functions sin θ and tan θ both approach the value of angle θ.



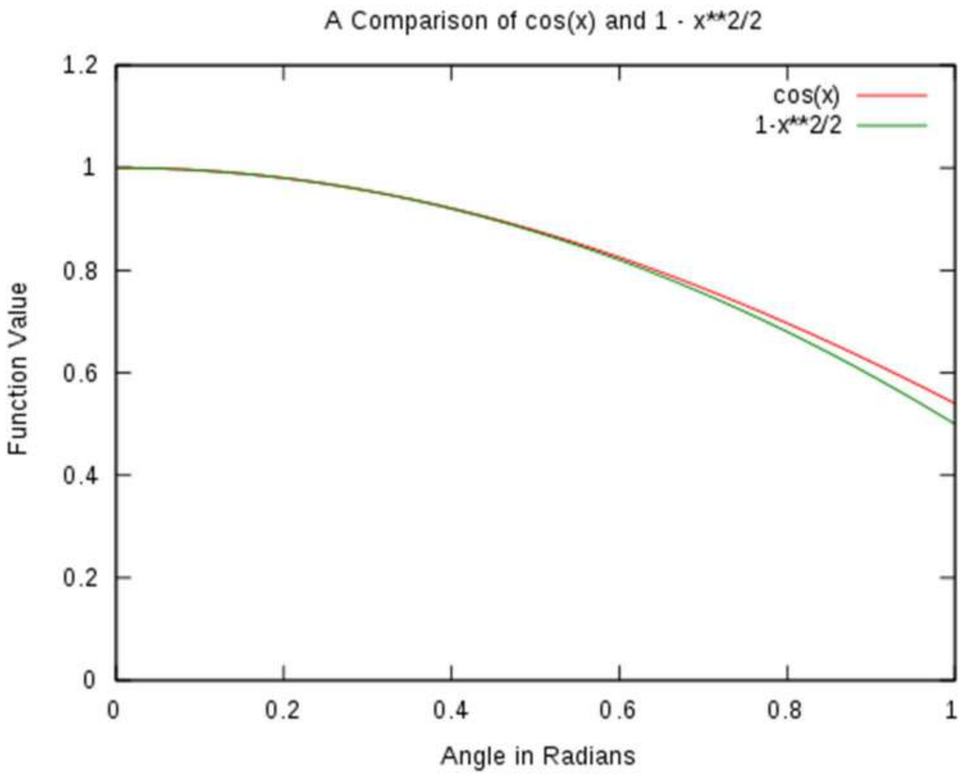
That is, for small angle θ:

$$\sin\theta \approx \theta$$

$$\tan\theta \approx \theta$$

Cos ϑ

As the angle θ approaches zero, the function cos θ approaches the value of the function $(1 - \theta^2/2)$:



That is, for small angle θ:

$$\cos\theta = 1 - \frac{\theta^2}{2}$$

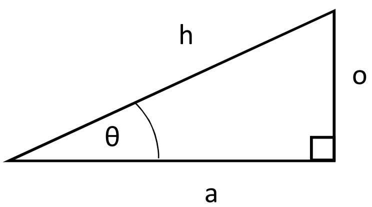
In many engineering applications, a mathematical model of a system can be significantly simplified if the small angle approximation can be justified. Investigate the range of angles for which the small angles approximations are valid. Angles must be in *radians*.

Note on units / dimensions

Be careful with respect to units (or generally, dimensions) when applying the small angle approximation. The trigonometric functions sine, cosine and tangent are ratios; they are dimensionless. When the trigonometric function is approximated using the small angle approximation, be careful not to change the dimensions.

For example:

The length of side o in the following diagram can be calculated if we know one angle and one other side length, e.g. angle θ and side h:



$$o = h\sin\theta$$

The result is of course a length. Now apply the small angle approximation:

$$o = h\sin\theta \approx h\theta$$

Note that $h\theta$ is being used as an *approximation* for $h \cdot \sin\theta$. We must retain the dimensions of $h \cdot \sin\theta$ [length x dimensionless = length] and not the dimensions of $h\theta$ [length x angle]. This example is quite obvious, but when the small angle approximation is used to simplify a more complicated calculation, it is easier to slip.

Circles

The basics of circle mathematics (radius, diameter, circumference, area) will already be well-known to you.

Curvature

Curvature (to be precise, *extrinsic curvature*) is the extent to which a curve differs from straightness (or a surface differs from being a flat plane). Curvature can be a scalar or vector quantity, but we will normally only be interested in the scalar quantity.

You should know how curvature is related to *radius of curvature*.

Periodic Motion

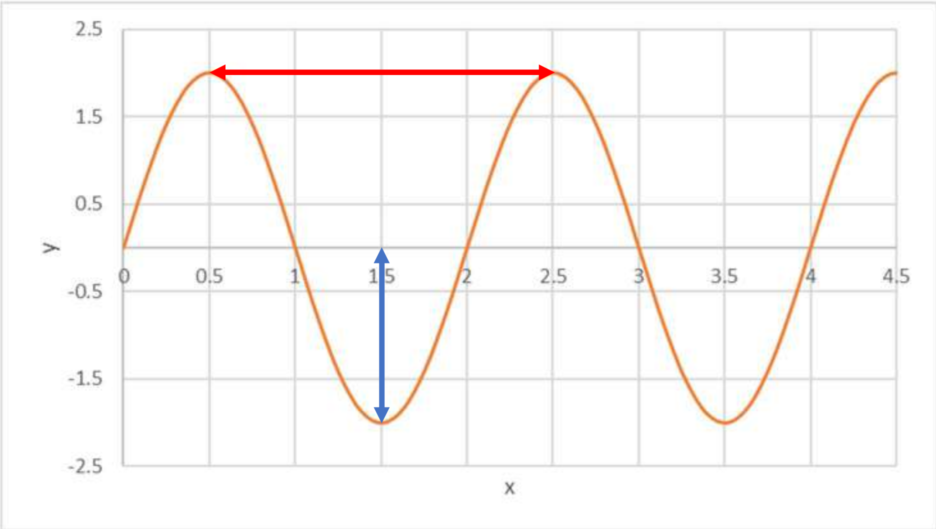
Periodic motion is a movement that repeats in a regular time interval. A sine wave is periodic. The full equation for a sine wave is:

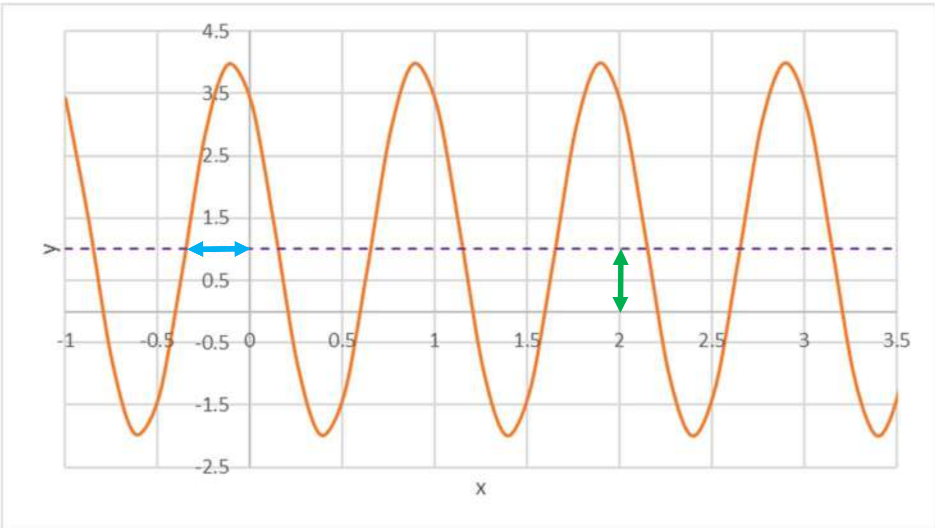
$$y = A\sin\big(B(x + C)\big) + D$$

Where:

A = amplitude
2π/B = period
C = phase shift (to the left)
D = vertical shift

Identify each of these parameters on the sine wave graphs below:





Frequency

Frequency (specifically *temporal frequency*) is the number of cycles per unit time. It is the inverse of period:

$$f = \frac{1}{T}$$

Where:

f = frequency
T = period

Simple Harmonic Motion

Simple harmonic motion (SHM) is a special case of periodic (or *oscillatory*) motion where the *restoring force* is directly proportional to displacement and in the opposite direction of the displacement.

A spring-mass system is an example of a simple harmonic oscillator. The restoring (spring) force is given by *Hooke's Law*:

$$F = -kx$$

Where:

F = spring force
k = spring stiffness
x = displacement

The negative sign indicates that the restoring force is in the opposite direction to displacement.

The frequency of a simple harmonic oscillator (*natural frequency*) is independent of the amplitude and initial phase. It depends only on the mass and *spring constant* (or *spring stiffness*):

$$\omega^2 = \frac{k}{m}$$

Where:

ω = angular frequency
k = spring stiffness
m = mass

Exercises:

1. What are the SI units for angular frequency?

2. What is the formula for *temporal* frequency (Hz)?

Calculus

Differentiation

Given an equation, or a *function*, we can *differentiate* to find the *derivative* of the function. The derivative function gives the gradient of the function at any point.

There are various notations used for differentiation, including the following:

	function	**derivative**
Leibniz's notation	y	$\frac{dy}{dx}$
Lagrange's notation	$f(x)$	$f'(x)$
Newton's notation	y	$\dot{y}$

Many functions fall into one of a few common and easily differentiated categories of functions. You should know the most common *standard derivatives* in the following table:

$\boldsymbol{y \text{ or } f(x)}$	$\boldsymbol{\frac{dy}{dx} \text{ or } f'(x) \text{ or } \dot{y}}$
ax^n	
$\sin ax$	
$\cos ax$	
e^{ax}	
$\ln ax$	
$\log_a x$	

Exercises:

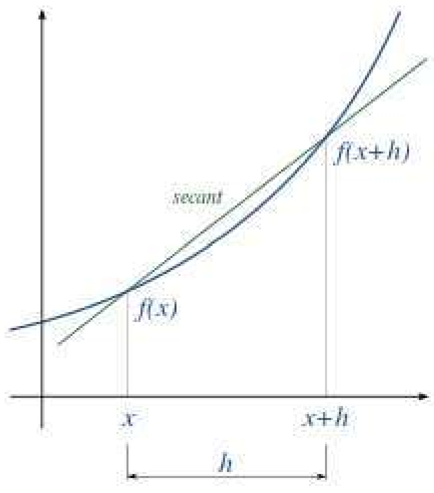
1. For the following function, find the co-ordinates of any stationary points and classify them as local maximum or minimum points:

$$y = x^3 - 6x^2 + 9x + 1$$

Numerical Approximation

In engineering we are often dealing with sets of real data rather than mathematical functions, so the techniques described above can't be employed. There are, however, methods to obtain approximate differentials which can be applied. Here we will look at the simplest category of numerical approximation methods, *finite difference formulas*.

Below, a *secant line* (a line that intersects a curve at two points) has been drawn through two points on the curve y=f(x), separated by a distance, h, along the x-axis.



The gradient of the secant line, $\frac{f(x+h)-f(x)}{h}$ is an approximation to the gradient of y=f(x) at x and the approximation improves as h becomes smaller. This method is called *forward differencing* or *one-sided differencing*. *Backward differencing* is the same technique but using a point at x-h instead.

Another finite difference method is *centred differencing*, where we take points either side of x, at x-h and x+h. Centred differencing is more accurate than one-sided differencing, however all of these methods are adversely affected by floating point errors when h is very small, so we can't simply keep on reducing the value of h to obtain better approximations, when using computers to run the calculations.

Exercises:

1. Write the formula for the approximate derivative of $y = f(x)$ using backward differencing. Use h to represent the small change in x.

2. Write the formula for the approximate derivative of $y = f(x)$ using centred differencing. Use h to represent the small change in x.

Integration

Integration, or *anti-differentiation*, is the reverse process to differentiation. When we differentiate a function, any constant terms are reduced to zero in the derivative, for example:

$$\frac{d}{dx}(x^3) = 3x^2$$

and:

$$\frac{d}{dx}(x^3 + 9) = 3x^2$$

Graphically, x^3+9 is simply x^3 shifted 9 units up the y-axis, so for any value of x, both functions have the same gradient and therefore the derivatives are the same.

However, this means that we must add a *constant of integration* when performing the reverse process to differentiation.

The notation for the *integral* of a function, y, with respect to x is:

$$\int y\, dx$$

For example, the integral of $3x^2$ with respect to x is:

$$\int 3x^2\, dx = x^3 + c$$

Where c is the constant of integration.

Like differentiation, many integrals fit into groups of common *standard integrals*. You should know the common *standard integrals* in the following table:

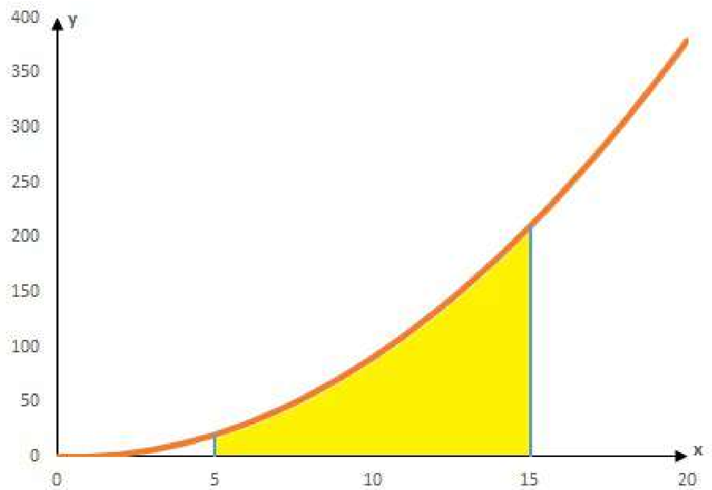
y	$\int y\,dx$
$k\ (a\ constant)$	
	$\frac{ax^n}{n+1} + c\ (except\ when\ n = -1)$
$\cos ax$	
$\sin ax$	
e^{ax}	

These examples are called *indefinite integrals*. *Definite integrals* have upper and lower boundaries which allow a definite value to be found.

The integral of a function represents the area under the curve of the function. The orange curve on the graph below shows the function:

$$y = (4 - x^2)$$

The yellow-shaded area is area between the function and the x-axis, between the bounds of x = 5 and x = 15.



This area under the curve can be found by integration of the function between the upper and lower bounds, denoted as follows:

$$\int_5^{15} (x^2 - x)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_5^{15} = \left(\frac{15^3}{3} - \frac{15^2}{2}\right) - \left(\frac{5^3}{3} - \frac{5^2}{2}\right) = 983\frac{1}{3}$$

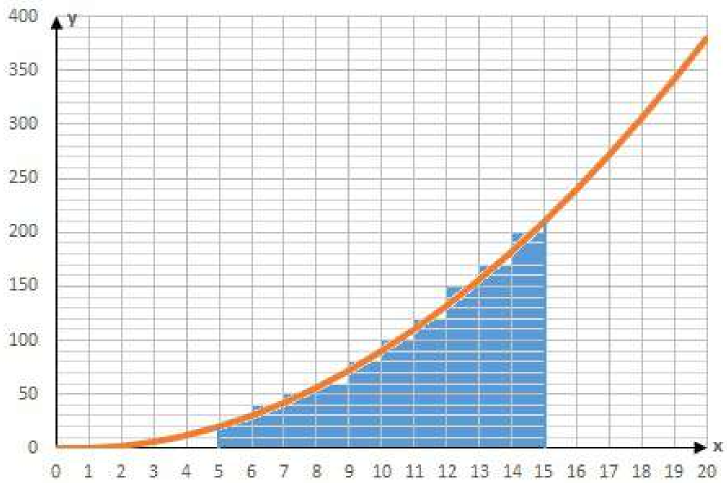
Note that the constant of integration appears in the integral evaluated at both the upper and lower bound, so it is cancelled when the lower integral is subtracted from the upper, so we can omit the constant when dealing with definite integrals.

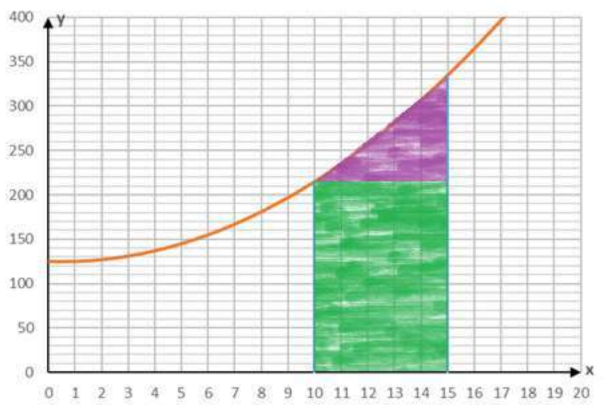
Numerical Integration

The previous section on integration covered just the very basics of *analytical integration* and some common types of integrals with simple solutions. Integration of other types of function can be much more difficult, and some functions can't be integrated analytically at all. In addition, we may want to integrate a set of measured data rather than a known algebraic function. In these cases, we can use numerical integration methods.

Manual Method

At the most basic level of numerical integration, we can simply count squares on a grid to find an enclosed area, or divide the area into convenient shapes.

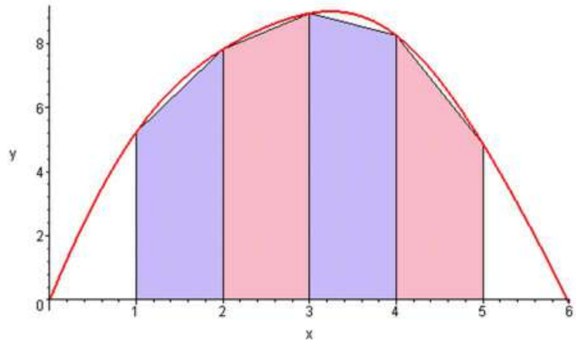




Clearly, these methods involve some degree of error. In the case of counting the squares, the error is simply a function of the size of the grid used – in theory we could get very accurate results using a very fine mesh. However, manual counting isn't very convenient…

Trapezium Rule

In the second example above, the triangle and rectangle combine to form a *trapezium* or *trapezoid* (a four-sided figure). We could improve accuracy by splitting the trapezium vertically in two or more strips, or *intervals*, as shown below. This method is called the *trapezium rule*.



For each strip, the area is:

$$\frac{\Delta x}{2}(y_i + y_{i+1})$$

where: Δx is the width of the strip and y_i and y_{i+1} are the y co-ordinates at either side of the strip.

Exercises:

1. Using the notation above, write the total area for four strips starting with y co-ordinates at y_0 to y_5.

2. From your answer above, write the general formula for n strips.

3. Use your formula to evaluate the integral of the following data set between x= 5 and x = 10

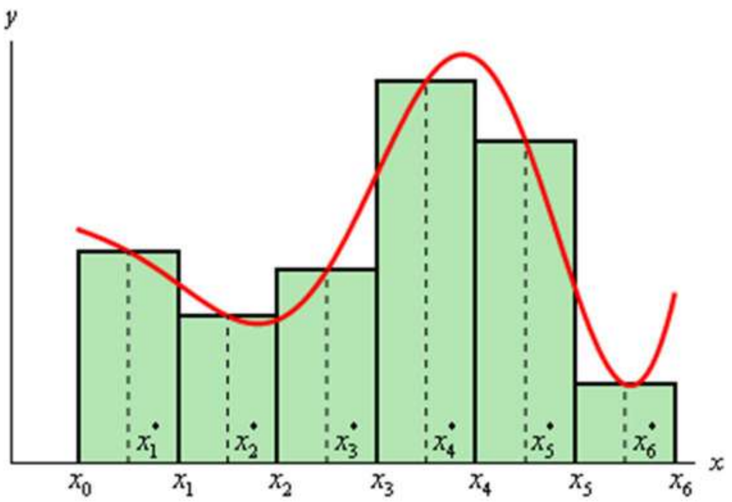
X	y
0	0.0
1	4.5
2	33.0
3	109.5
4	258.0
5	502.5
6	867.0
7	1375.5
8	2052.0
9	2920.5
10	4005.0

4. The data set above actually comes from the function $y = 4x^3 + \frac{x}{2}$. Evaluate the definite integral of this function between x = 5 and x = 10, then find the percentage error in the value returned by the trapezium method.

5. What is the principal influence on the amount of error of the trapezium rule calculation?

Midpoint Rule

The midpoint rule is conceptually similar to the trapezium rule, dividing the area to be approximately integrated into strips each having a simple geometric shape, the area of which is easily calculated. However, instead of trapezia constructed between adjacent data points, the midpoint rule places a rectangular strip cantered on the data point:



Exercises:

1. Write a formula for the area, A of a single strip with midpoint at (x_1, y_1) and width, Δx.

2. Extend the formula for 4 strips of equal width, Δx, with midpoints at y_1 to y_4.

3. Write the general formula for the approximate integral, A, for evenly spaced strips of width Δx and midpoints at y_1 to y_n.

Simpson's Rule

Another common numerical integration method is Simpson's Rule, which uses parabolic approximations to the data or curve to be integrated, each of which spans three data points (or two strip-widths. Although this sounds much more complicated, the formula actually reduces very simply to:

$$area = \frac{\Delta x}{3}(F + L + 4E + 2R)$$

Where:

Δx = the strip width
F = the **F**irst y co-ordinate
L = the **L**ast y co-ordinate
E = the sum of the **E**ven-numbered co-ordinates (i.e. the 2nd, 4th etc.)
R = the sum of the **R**emaining, odd-numbered y co-ordinates (n.b. we avoid writing an "O" here to avoid confusion with a "0" (zero)…)

Clearly, Simpson's rule can only be applied when the number of intervals is even (i.e. the number of data points is odd).

Mechanics

Kinematics: Equations of Motion

These are the standard formulas relating distance, speed, acceleration and time for a body with *constant acceleration*.

$$v = v_0 + at$$

$$r = r_0 + v_0 t + \frac{1}{2}at^2$$

$$r = r_0 + \frac{1}{2}(v + v_0)t$$

$$v^2 = v_0^2 + 2a(r - r_0)$$

$$r = r_0 + vt - \frac{1}{2}at^2$$

Where:

r_0 = the particle's initial position
r = the particle's final position
v_0 = the particle's initial velocity
v = the particle's final velocity
a = the particle's acceleration
t = the time interval

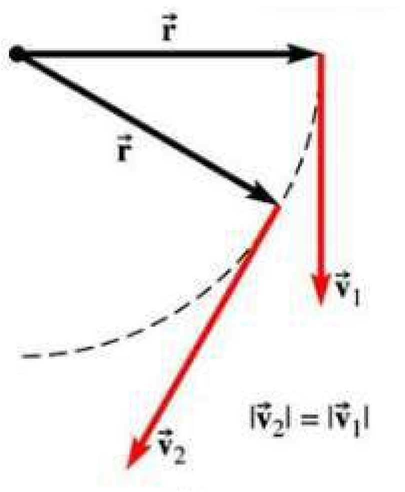
Further study:

https://physics.info/motion-equations/

https://physics.info/kinematics-calculus/

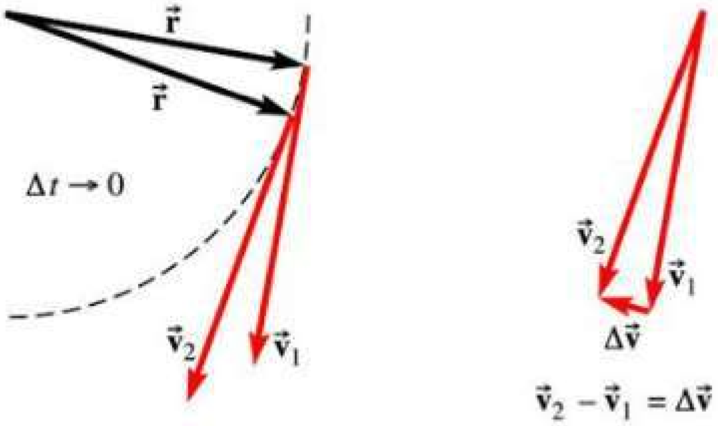
Kinematics: Circular Motion

The sketch below represents a particle on a circular path at constant angular velocity. The velocity vectors at time t_1 and time t_2 are shown.



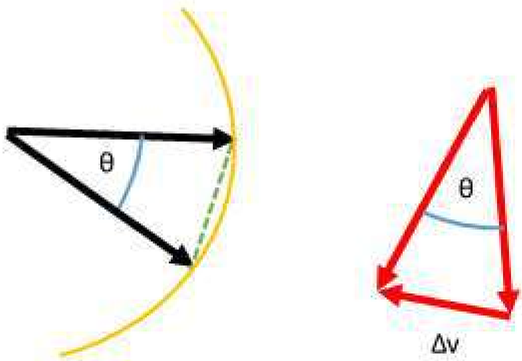
The velocity is changing: the magnitude of the velocity remains the same, but the direction varies as the body moves around its circular path. Since there is a change in velocity in the time interval between t_1 and t_2, we know there must be an ______________________.

Below we see that as the time interval $t_2 - t_1 = \Delta t$ approaches zero, the change in velocity $\mathbf{v_2} - \mathbf{v_1} = \mathbf{\Delta v}$ (all vectors, of course) approaches the perpendicular of the velocity vector.



Remembering that the instantaneous velocity is tangent to the circular path, the acceleration perpendicular to the velocity is therefore towards ____.

Notice that the angle between the (black) position vectors and the angle between the (red) velocity vectors in the sketch above, is the same. These vectors are reproduced below, with the angle exaggerated for convenience.



Since v = ωr the triangles are *similar* (i.e. proportional), so:

$$\frac{\Delta v}{v} = \frac{c}{r}$$

Where c is the length of the *chord* between the ends of the position vectors (green dashed line) and v is the magnitude of the linear velocity. As Δt→0, the actual distance travelled, s, around the circular path in the time interval tends towards the *chord* length:

$$\frac{\Delta v}{v} = \frac{s}{r}$$

And the distance travelled, s, in the time interval Δt is equal to the magnitude of the velocity multiplied by the time interval:

$$\frac{\Delta v}{v} = \frac{v\Delta t}{r}$$

$$\rightarrow \frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

The *centripetal acceleration* is therefore:

$$\rightarrow \; a = \frac{v^2}{r}$$

So, we know that circular motion corresponds with an acceleration perpendicular to the instantaneous velocity, i.e. towards the centre of the circle, and that the magnitude of the acceleration is v^2/r.

Newton’s Laws

Motorsport engineering is concerned with objects and speeds that fall well within the bounds of *classical* or *Newtonian* mechanics. You should know and be comfortable applying Newton’s laws of motion:

First law:	In an inertial frame of reference an object remains at rest or constant velocity unless acted upon by a force	$\sum \boldsymbol{F} = 0 \leftrightarrow \frac{d\boldsymbol{v}}{dt} = 0$
Second law*:	In an inertial frame of reference, the sum of forces acting on an object is equal to the mass of the object multiplied by the acceleration of the object	$\boldsymbol{F} = m\boldsymbol{a}$
Third law:	When one object exerts a force, F_A, on a second object, the second object simultaneously exerts a force, F_B, of equal magnitude and opposite direction on the first object	$\boldsymbol{F}_A = -\boldsymbol{F}_B$

* The second law actually states that the rate of change of *momentum* (mass x velocity) is equal to the applied force. The law is only applicable to constant-mass systems, so the rate of change of momentum is equal to mass x acceleration:

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = \frac{d(m\boldsymbol{v})}{dt} = m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{a}$$

Where **p** is momentum [kg·ms^{-1}]

Note that, although in some calculations the mass is non-constant (e.g. vehicle mass decreases as fuel is consumed), the rate of change of mass is usually sufficiently low that constant mass can be assumed in the instant.

It is important to understand the condition, applicable to the first and second laws, of an *inertial reference frame*. This is practically demonstrated through free body diagrams and D’Alembert’s principle.

Free Body Diagrams & D'Alembert's Principle

Free Body Diagrams (FBDs) are a crucial step in analysing many physical systems. An FBD shows only the body to be analysed and the *external* forces acting *on* the body. If the body interacts with other bodies or constraints, these interactions are shown on the FBD as external forces acting on the body. Forces exerted *by* the body on its environment, and forces *internal* to the body, are not shown. See https://en.wikipedia.org/wiki/Free_body_diagram or another resource to review FBDs in more detail.

Exercises:

Draw an FBD showing a driven wheel moving up a slope, ensuring all relevant forces and torques are shown and act in the correct direction.

Free, sliding and fixed vectors

Force (and other vector quantities) can be classified as *free*, *sliding* or *fixed*. Identifying which forces fall into which categories is useful in solving force systems in FBDS.

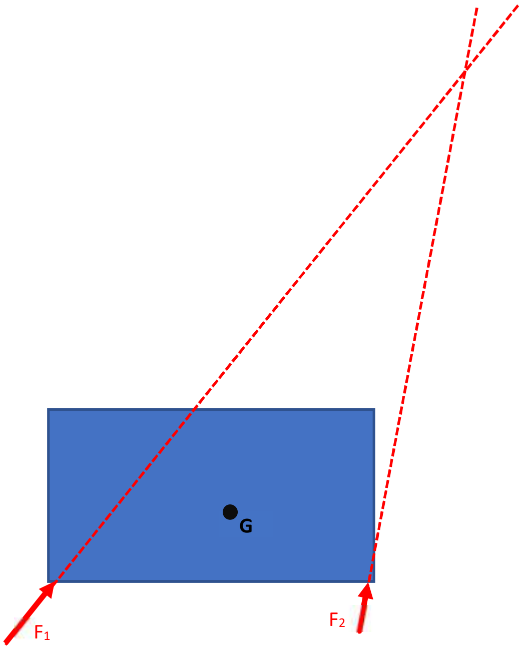
Free vectors

A vector is defined by its magnitude and direction. It is "free" to move in space. For example, when we place force vectors head to tail to find the resultant, we move the vectors freely in order to place them head to tail. The velocity vector of a body in translational motion is a free vector: it is not constrained to a certain point but applies to all points in the body.

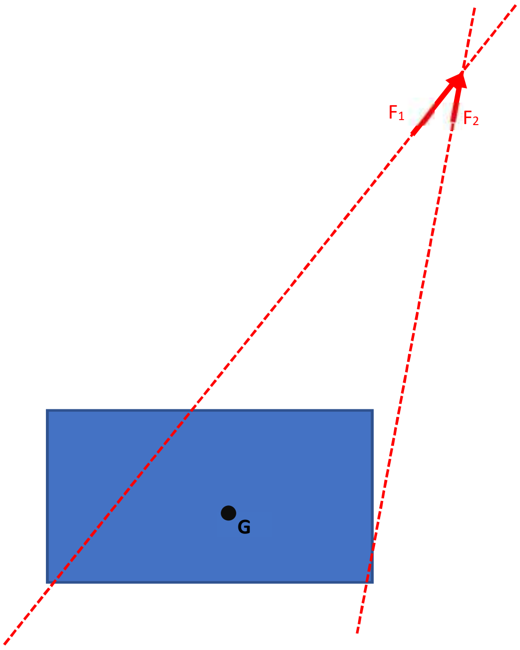
Sliding vectors

These vectors are constrained to a *line of action*. The effect of a force applied to a *rigid* body does not change if we slide the force anywhere along its line of action. In many applications in race engineering, the bodies can be considered rigid, allowing the forces acting on the body to be applied anywhere along their lines of action. This can be very useful in analysing systems. The following example shows forces on a rigid body being slid along their lines of action to find the resultant, along with a graphical method for determining the magnitude and direction of the resultant:

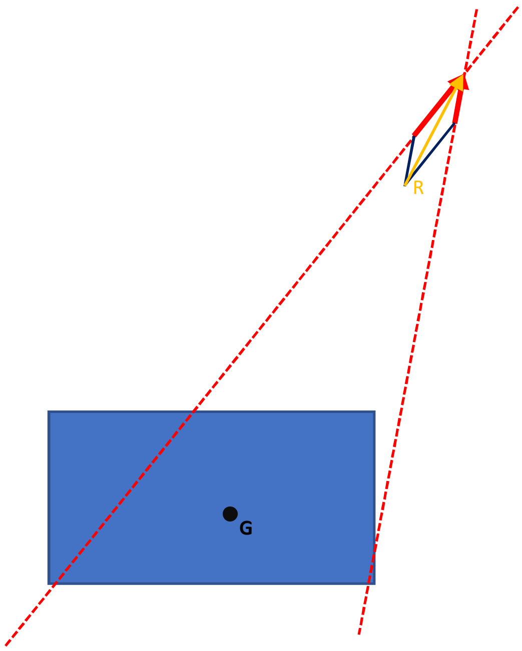
Two forces acting on a rigid body. Point G represents the centre of gravity.



The lines of action of the two forces are extended until they cross.

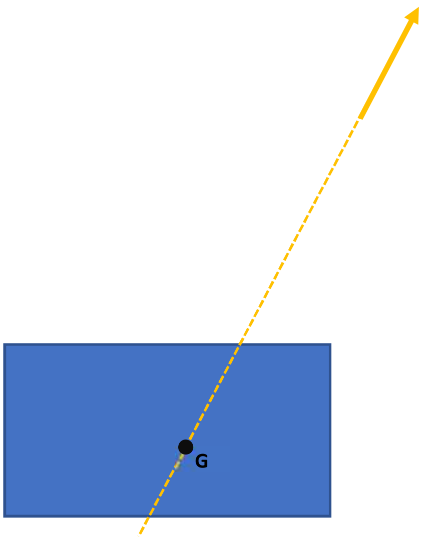


Slide the forces along their lines of action such that the heads of the vectors meet at a point.



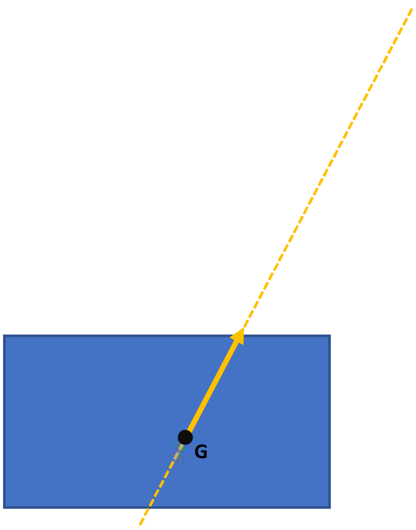
Parallelogram method*: complete the parallelogram formed on two sides by the two forces; the resultant force is the diagonal of the parallelogram with the head of the vector at the same point as the two component forces.

*Make sure you know the parallelogram method for finding the resultant of two concurrent forces, it can be a useful graphical/conceptual method in addition to algebraic methods of calculation.



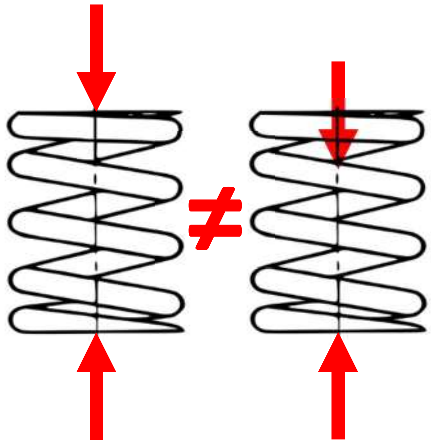
Extend the line of action of the resultant.

Slide the resultant along its line of action. In this case, it passes through the centre of gravity, i.e. the forces on the body cause a linear acceleration and do not cause angular acceleration.



Fixed vectors

When a force is applied to a deformable body, it is constrained to act at a single point of application. For example, we can't slide a force acting on a coil spring along its line of action as that would change the effect of the force on the spring.

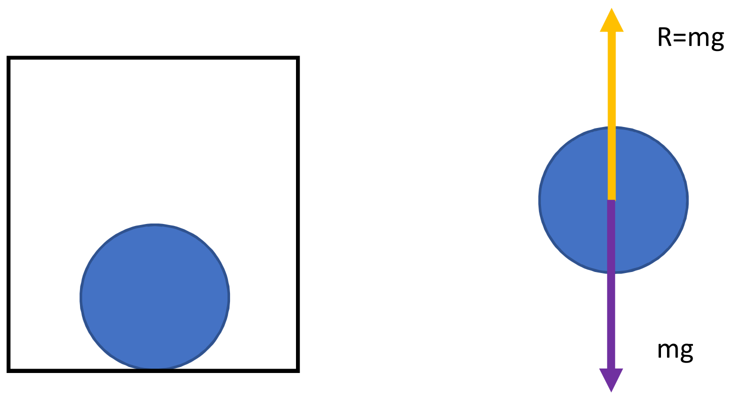


D'Alembert's Principle

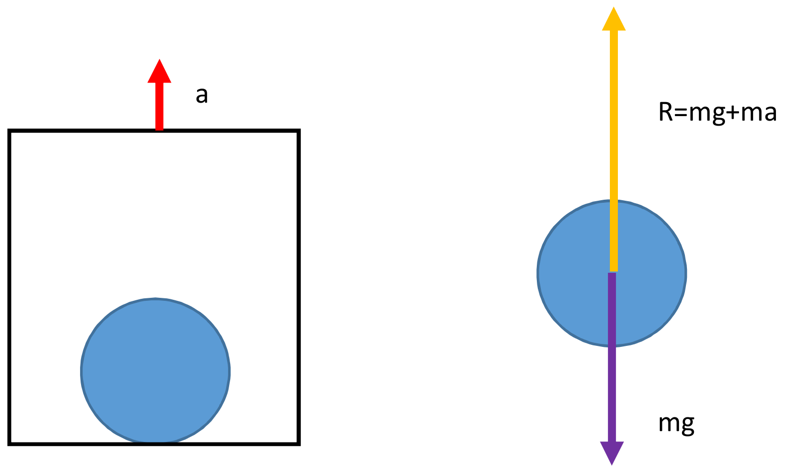
In some situations, it can appear like Newton's Laws don't correctly define the motion of a body in motion. For example, a passenger in a car that is braking feels pushed forward relative to the car and might therefore say that there is a forward force acting on their body due to its inertia. However, if we try to draw this situation in a free body diagram, we would have the braking acceleration acting backwards (since we know the car and its passenger are decelerating) and the forward-acting force experienced by the passenger. If we continue and ascertain the magnitude of these forces, we find that the inertial force is equal to the braking force, so the opposing forces have no resultant and the passenger is either stationary or travelling at constant speed, which is clearly not the case under braking.

That forward inertial force was never actually there. Similarly, the *centrifugal* force experienced on a carousel or in a cornering car and the *Coriolis* force that results in circular motion of water draining from a bath, are not actually there. In fact, these are examples of *pseudo forces* (also known as *fictitious forces* or *d'Alembert forces*), which are very useful concepts, when correctly applied.

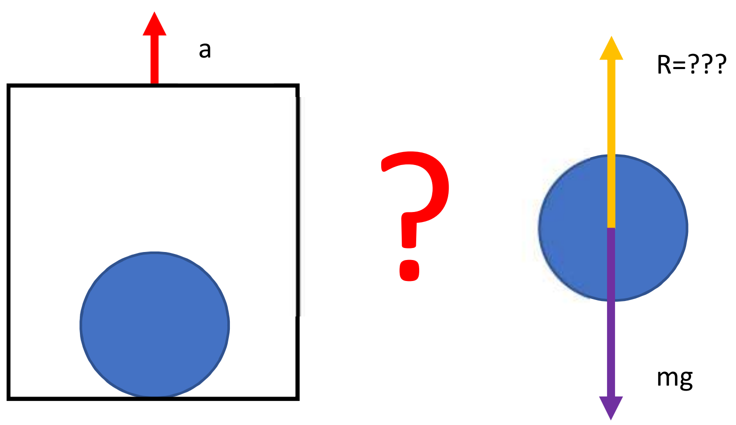
In the sketch below, the ball sits in a stationary lift (or elevator). The only forces acting on the ball are that of gravity and the reaction from the floor of the lift. The two forces are equal and opposite, so the net force on the ball is zero and the ball remains stationary. The free body diagram is shown on the right.



In the next diagram, the lift is accelerating upwards with acceleration, a.

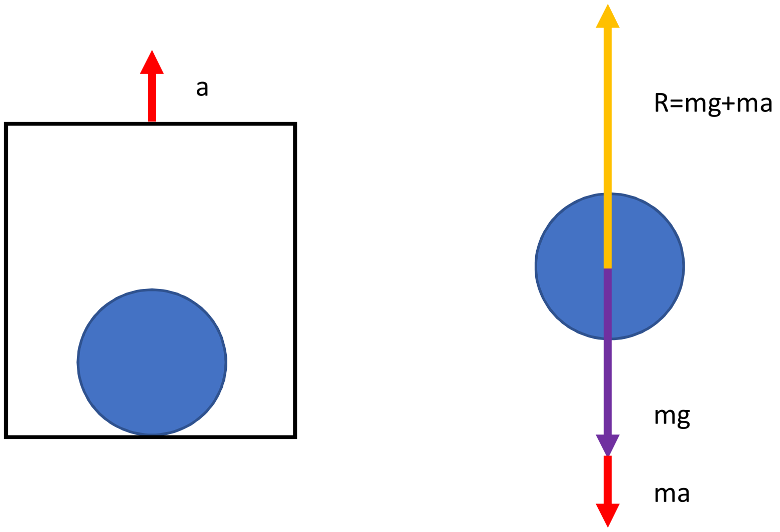


The lift exerts a greater reaction force on the ball and the ball accelerates with the lift. This is the situation as seen by a stationary observer (stationary on the ground, not moving in the lift). However, to an observer inside the lift, the ball appears stationary, as if the gravitational and reaction forces remain balanced. It appears to the moving observer that Newton's Laws do not apply.



The difference is that the observer in the lift is subject to the acceleration of the lift. Newton's Law apply in a Galilean / inertial reference, i.e. relative to the Earth (the stationary observer). When the observer (or the reference frame) is accelerating, Newton's Law do not hold.

D'Alembert's principle adds a *fictitious force* equivalent to the *inertia* of the mass. This allows us to use Newton's Laws in an accelerating frame:

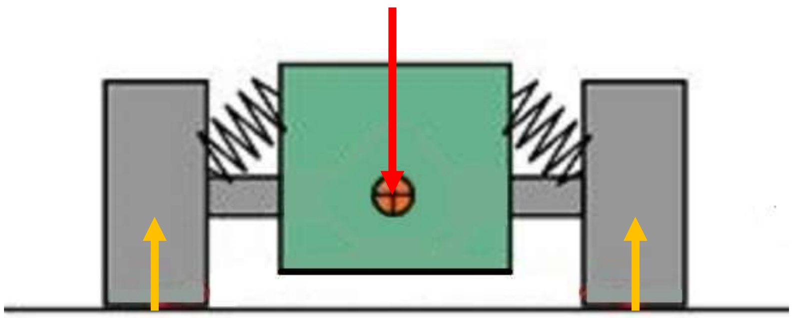


The forces are now balanced, and the behaviour of the ball is consistent with what the observer (accelerated) in the lift sees.

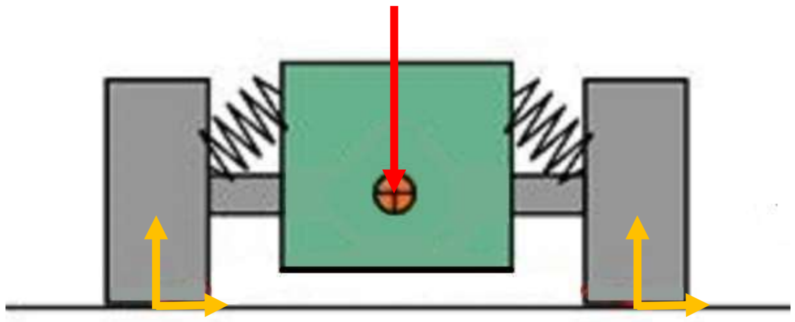
This manipulation is used often to solve dynamic problems, but it is easy to wrongly add inertial forces when they should not be included, so an awareness of d'Alembert's Principle is useful in applying inertial forces only when it is appropriate.

The benefit of d'Alembert's Prinicple is that it allows a dynamic problem to be turned into a (simpler) static problem. A common example in vehicle dynamics is that of total load transfer.

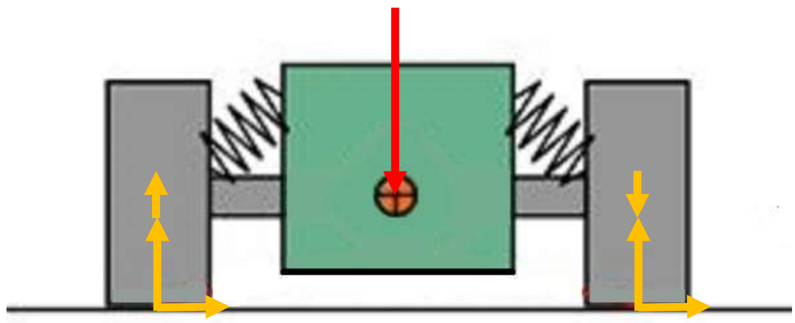
When static, the car is subject to a gravitational force (weight) and a reaction at each contact patch. The same forces apply in the lateral cross-section shown (below) when the car is running straight ahead.



When the car turns (right in the example), lateral forces are developed at the contact patches (below). As shown in the diagram, the car is now clearly not in equilibrium – there is an acceleration towards the right. There is now also an unbalanced moment due to the tyre lateral forces.

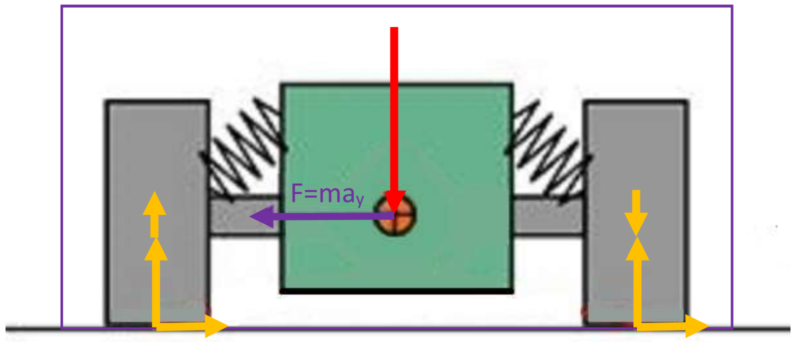


At steady state, this moment is balanced by load transfer – the total reaction at the left wheel is increased and the total reaction at the right wheel is decreased, compensating the "roll" moment of the tyre lateral forces (below).



To simplify analysis, we can draw a frame around the car, fixed relative to the centre of gravity (CoG) of the car. Since the car is accelerating towards the right, the frame is accelerated the same. Therefore, the car is "stationary" within the frame. We must add a (fictitious) d'Alembert/inertia force to compensate the acceleration of the reference frame, having the same acceleration as the frame and the opposite direction.

n.b. the following equations are written such that the sign term in the equation corresponds to the direction of the arrow in the accompanying sketch (the numerical value of the forces/moments therefore represent only the intensity and the positive direction is that of the diagram arrow. This is a common approach in simple analyses, just need to be aware of the method used and remain consistent.



Since the car is stationary, forces must be in equilibrium (ΣF=0). From the diagram above, we can write the force balance:

$$\sum F_z = R_L + \Delta R_L + R_R - \Delta R_R - mg = 0$$

$$\sum F_y = F_{yL} + F_{yR} - ma_y = 0$$

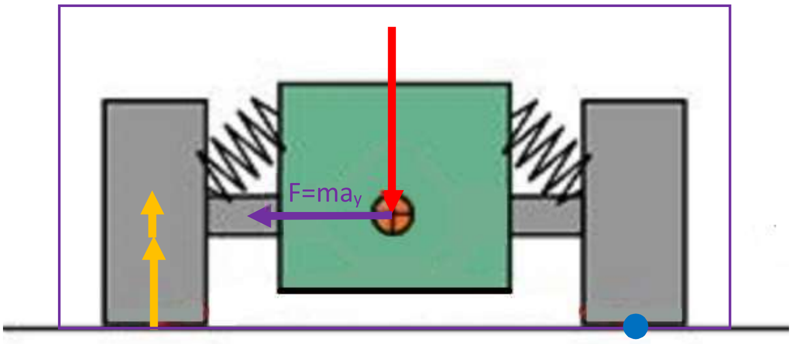
Looking at the actual static equilibrium of the vehicle (first diagram), we can see that:

$$\sum F_{z-static} = R_L + R_R - mg = 0 \;\rightarrow\; R_L + R_R = mg$$

The dynamic balance therefore simplifies to:

$$\sum F_z = \Delta R_L - \Delta R_R = 0$$

The moments must also be in equilibrium (ΣM=0) in the reference frame. The other advantage of having reduced the dynamic problem to a static problem is that we *can choose any point about which to balance the moments*, since the moment about any point must be zero. We can choose a point that lies on the line of action of the lateral tyre forces (i.e. the ground plane) and that is also on the line of action of one of the vertical forces, thereby reducing the number of unknowns. Below, the moments are taken about the right-side contact patch, anti-clockwise +ve. For demonstration purposes, all the forces that pass through the moment axis (M_i=0) have been removed from the diagram. The CG is assumed to be a mid-track.



The moment balance about this point is then:

$$\sum M = ma_y h_{CG} + \frac{mgT}{2} - R_L T - \Delta R_L T = 0$$

Again, looking back at the actual static moments about the same point:

$$\sum M_{static} = \frac{mgT}{2} - R_L T = 0$$

The dynamic balance therefore simplifies to:

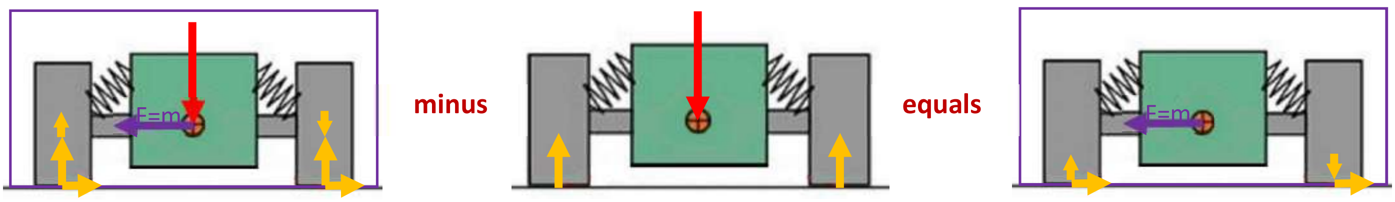
$$\sum M = ma_y h_{CG} - \Delta R_L T = 0$$

Since the static left/right reaction forces simplify out of the force and moment balance equations, we can conclude that static weight distribution has no influence on load transfer. From the equation above:

$$\Delta R_L = \frac{ma_y h_{CG}}{T}$$

This is the amount of load transfer from inside wheels to outside wheels (at steady state). We can conclude that the only parameters that influence the amount of load transfer are the vehicle mass, lateral acceleration, CG height and vehicle track. Study of tyre performance characteristics will show that load transfer is to be minimised for best performance, hence performance / race cars have the minimum mass, lowest CG and widest track possible within the constraints of regulations, resources/budget and other considerations (e.g. compromise between wide track for load transfer and narrow track for increased corner radius).

Note that the simplification of the equations could also have been carried out at the diagram level:



The static forces have no influence on load *transfer* and we can move directly to study a diagram showing only the dynamic (steady state) forces shown; the result is the same.

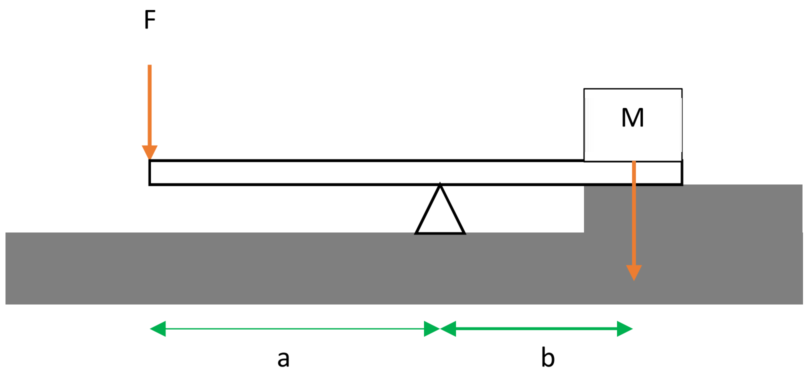
Levers

A *lever* is a simple machine that can change the magnitude and line of action of a force. To analyse levers, we can apply the principle of moments:

The sum of the clockwise moments about any point must be equal to the anticlockwise moments about that point*

*or *counter-clockwise*

For example, to find the force, F, required to *just* lift the mass, M, in the system below:



We can write the moments of the two forces (F and the weight, Mg, of the mass):

$$clockwise\ moments = anticlockwise\ moments$$

$$\rightarrow \quad Mg \times b = F \times a$$

Then make F the subject:

$$\rightarrow \quad F = \frac{Mgb}{a}$$

We can also define a *force ratio*, which is simple the ratio of the equilibrium forces:

$$force\ ratio = \frac{load\ force}{effort\ force} = \frac{Mg}{F} = \frac{a}{b}$$

This shows that the ratio of loads in the inverse of the ratio of lever arm lengths. Another useful ratio is the movement ratio, which we find in the definition of efficiency of a simple machine:

$$efficiency\ [\%] = \frac{force\ ratio}{movement\ ratio} \times 100\%$$

If we assume no losses due to friction etc., then for 100% efficiency:

$$movement\ ratio = force\ ratio = \frac{a}{b}$$

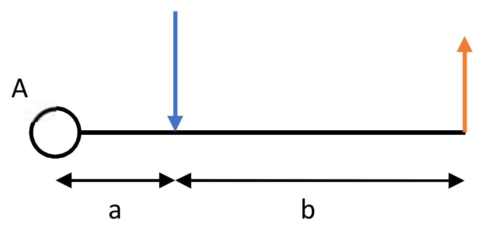
Since we are using moments, it is important to note that the lever arm length is the perpendicular distance between the fulcrum (pivot point) and the line of action of the force.

Exercises:

All answers should include a diagram (unless already provided).

1. A lever of total length 2m has a fulcrum point at 0.5m from the load. What is the minimum effort required to just lift a load of 0.75 kN assuming no losses?

2. The sketch below represents a lever arm that pivots about point A. The orange arrow represents the effort and the blue arrow is the load. The load acts at 30cm from the pivot and the effort at 75cm from the load. Assume 100% efficiency.



   a. Calculate the force ratio

   b. Calculate the movement ratio

   c. What force is required to balance a load of 1500 N?

Solid Mechanics / Materials

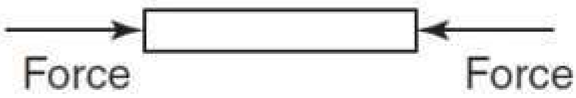
All forces applied to an object cause a change in the motion or shape of the object. In this section, we will be review changes in shape due to forces.

Tensile and Compressive Forces

Tension is a force that has a *stretching* effect on an object:

A *tensile force* (a force that results in tension) causes the length of the object in tension to increase. Note that there must be a force at both ends of the object for there to be tension.

Compression is a force that has a *squeezing* or *crushing* effect on an object:



A *compressive force* (a force that produces compression) will act to decrease the length of the object in compression. Again, there must be an opposing force.

Exercises:

Complete the following statements:

1. A rubber belt stretched between two pulleys is in

   ____

2. A steel pillar supporting a bridge is in

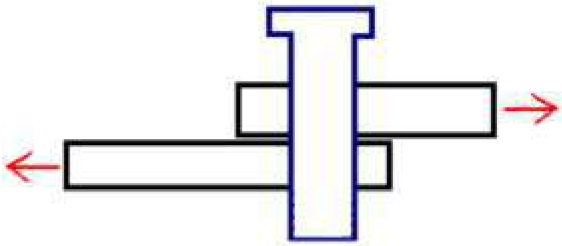
   ____

3. A horizontal beam supported at each end is in

   ____

Shear Force

Shear is a force that causes a sliding effect between adjacent surfaces of a material. Shear occurs similarly to tension and compression, however in the case of shear, the opposing forces are *unaligned*:



Shear forces cause bending, sliding or twisting motions.

Exercises:

Give three examples of vehicle components in shear, with a simple diagram for each:

1.

2.

3.

Stress

Whenever forces act on an object or material to cause a change in its shape, or *deformation*, the material is under *stress*. Stress is the ratio of the force applied to the perpendicular cross-sectional area of the material:

$$\sigma = \frac{F}{A}$$

Note that stress has the same dimensions and units as pressure, pascals Pa or N/m^2.

In the case of tensile or compressive forces, the cross-sectional area is the area perpendicular to the direction of the force; for shear forces the area is parallel to the direction of force. Stress is usually represented by the Greek letters sigma (σ) for tensile and compressive stress and tau (τ) for shear stress.

Exercises:

1. A rectangular bar of cross-sectional area 75 mm^2 is subjected to a tension of 15 kN. Calculate the stress in the bar.

2. A round wire has a tensile force of 60.0 N applied to it, which produces a stress of 3.06 MPa in the wire. What is the diameter of the wire?

3. A 5 mm diameter bolt is loaded in shear such that the shear stress in the bolt is 120 MPa. Determine the value of the shear force applied.

4. A square-section support of side 12 mm is loaded with 10 kN compression. Find the compressive stress in the support.

Strain

The change in dimension of an object or material subject to a force is called *strain*.

For a material in tension or compression, the strain is the ratio of the change in length to the original length prior to application of the load and is represented by the Greek letter epsilon (ε):

$$\varepsilon = \frac{x}{L}$$

Note that since strain is a ratio of two lengths, it is dimensionless. As such, it is often expressed as a percentage.

For a material in shear, the strain is represented by gamma (γ) and the change in length, x, is defined as shown in the diagram below:

$$\gamma = \frac{x}{L}$$

Exercises:

1. A 1.60 m bar contracts along its axis by 0.1 mm when subjected to a compressive force. Calculate the strain as a percentage.

2. A wire of unloaded length 2.50 m has a strain of 0.012% when subjected to a tensile force. Determine the extension of the wire when loaded.

3. A hole of diameter 50 mm is to be punched in a 2 mm thick metal plate. The required shear stress is 500 MPa. Determine:
    a. The minimum force that must be applied to the punch.

    b. The compressive stress in the punch at the value found in (a).

Elasticity and Hooke's Law

The *elasticity* of a material is its ability to return to its original shape and size when external forces are removed. Up to certain load limits, rubber, steel and polythene are examples of elastic materials. When a gradually-increasing load is applied to such materials, the extension of the material will generally follow a linear relationship with the applied force:

Note that, in the context described above, the axes are inverted with respect to the conventions described earlier in this module ("input" along the x-axis). The reason for this will become apparent shortly.

This linear relationship will break down above a certain load. The point where extension is no longer directly proportional with the applied load is called the *limit of proportionality*. Beyond this point, the material may exhibit a non-linear elasticity over a small range, up to the *elastic limit*.

Beyond the elastic limit, the material no longer returns to its original size and shape when the load is removed. A material that does not return to its original shape when an applied load is removed exhibits the property of *plasticity*. Thus an elastic material subjected to a load above its elastic limit behaves as a plastic material.

Hooke's Law is the description of a material in its elastic phase up to the limit of proportionality:

Within the limit of proportionality, the extension of a material is proportional to the applied force.

$$x \propto F$$

That is, the force applied is equal to the extension multiplied by a constant of proportionality, typically denoted k:

$$F = kx$$

This constant, k, is referred to as the *stiffness* of the material.

Coil Spring Stiffness

The stiffness (or spring rate), k, of an ideal coil spring (or *helical* spring) is given by:

$$k = \frac{Gd^4}{8nD^3}$$

Where:

G = modulus of rigidity (or *shear modulus*) of spring material
d = wire diameter
n = number of active coils
D = mean diameter (= total spring diameter – wire diameter)

Exercises:

1. What is a typical, approximate value of shear modulus for the steel used in suspension coil springs?

Torsion Bar Stiffness

Suspension systems often incorporate torsion bars as well as, or instead of, helical springs. Torsion bars may be used as anti-roll bars or as main suspension springs. The torsional stiffness of a beam (with uniform cross-section over its length) is:

$$\frac{T}{\theta} = \frac{GJ}{L}$$

Where:

T = applied torque
θ = angle of twist
G = modulus of rigidity (shear modulus) of material
J = torsional constant
L = beam length

The torsional constant is a geometric property of the beam and is identical to the *second moment of area* of the beam normal to its cross-section, i.e. the *polar moment of inertia*. For a solid beam of circular cross-section:

$$J = \frac{\pi D^4}{32}$$

Where:

D = outside diameter of beam

For a hollow circular beam:

$$J = \frac{\pi(D^4 - d^4)}{32}$$

Where:

D = outside diameter of beam
d = inside diameter of beam

Exercises:

1. Noting that a helical spring is simply a torsion bar wound into a coil, explain the effect on spring stiffness of doubling the length of a helical spring.

Rotational Dynamics

Rotational dynamics can be considered as analogies to Newtons laws for translational motion:

	Translational		Rotational	
First law:	An object remains at rest or at constant velocity unless acted on by an external force	$\sum \boldsymbol{F} = 0 \leftrightarrow \frac{d\boldsymbol{v}}{dt} = 0$	An object remains at rest or at constant angular velocity unless acted on by an external torque	$\sum \boldsymbol{\tau} = 0 \leftrightarrow \frac{d\boldsymbol{\alpha}}{dt} = 0$
Second law:	The net external force acting on an object is equal to the mass of the object multiplied by the acceleration of the object	$\boldsymbol{F} = m\boldsymbol{a}$	The net external torque acting on an object is equal to the moment of inertia of the object multiplied by the angular acceleration	$\boldsymbol{\tau} = I\boldsymbol{a}$
Third law:	When one object exerts a force, F_A, on a second object, the second object simultaneously exerts a force, F_B, of equal magnitude and opposite direction on the first object	$\boldsymbol{F}_A = -\boldsymbol{F}_B$	When one object exerts a toque, τ_A, on a second object, the second object simultaneously exerts a torque, τ_B, of equal magnitude and opposite direction on the first object	$\boldsymbol{\tau}_A = -\boldsymbol{\tau}_B$

Torque (or *moment* or *moment of force*) is the tendency of a force to rotate a body to which it is applied. It is a *pseudovector* since the direction of the vector is defined by convention.

Moment of inertia, I, is the resistance of a body to angular acceleration. For a point mass:

$$I = mr^2$$

Where:

m = mass
r = distance from axis of rotation

The moment of inertia of any body is the sum of the moments of inertia of all the point masses of the object. Simple formulas for the moment of inertia of standard, simple objects can be looked up in tables. The moment of inertia of more complex objects may be constructed by splitting up the objects into simpler components and employing the *parallel axis theorem*.

Fluid Mechanics

Reynold’s Number

The Reynold’s number, Re, is a dimensionless quantity used to predict fluid flow behaviour. It represents the ratio of inertial forces to viscous forces within the fluid.

$$Re = \frac{\rho V L}{\mu}$$

Where:

ρ = fluid density
V = velocity
L = characteristic length*
μ = (dynamic) viscosity

* this length is selected by convention, according to the application.

Exercises:

1. What is the upper limit of Reynold’s number for *laminar* flow?

2. Above what value of Reynold’s number will fluid flow be *turbulent*?

Bernoulli’s Principle

Bernoulli’s principle relates fluid velocity to pressure. In simplified form, Bernoulli’s equation is:

$$p + q = p_0$$

Where:

p = pressure*
q = dynamic pressure
p_0 = total pressure

* the pressure is sometimes referred to as *static pressure* in order to differentiate from the total pressure.

That is, the total pressure is equal to the sum of static pressure and dynamic pressure. This total pressure is constant along a streamline.

Exercises:

1. What is the formula for dynamic pressure?

2. Substitute the formula for dynamic pressure into the simplified form of Bernoulli's equation (above) and use this to show the relationship between flow velocity and pressure.

Pascal's Law

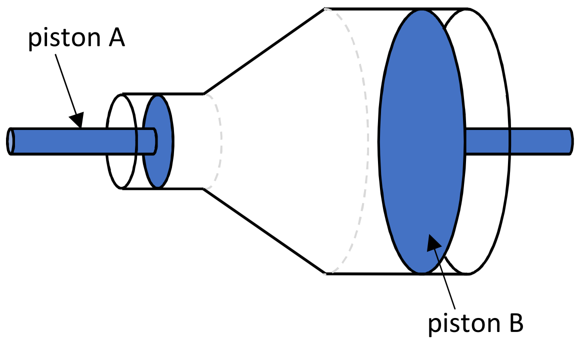
A change in pressure at any point in an enclosed fluid at rest is transmitted undiminished to all points in the fluid:

$$\Delta P = \rho g(\Delta h)$$

Where:

ΔP is the *hydrostatic pressure* [Pa]
ρ is the fluid density [kg/m^3]
g is the acceleration due to gravity [m/s^2]
Δh is the difference in height between the two points of interest within the fluid

This is the principal utilised in hydraulic brakes. Pushing on the brake pedal forces a piston (A) against the brake fluid, increasing the pressure in the fluid. The pressure increase is transmitted throughout the fluid volume. At the other end of the system, the fluid acts against another piston (B). The effect of any difference in height in a braking system is negligible, therefore:



$$P_A = P_B$$

Pressure, by definition, is (perpendicular) force per unit area, so:

$$P_{A=B} = \frac{F_A}{A_A} = \frac{F_B}{A_B}$$

$$\rightarrow \quad F_B = F_A \frac{A_B}{A_A}$$

That is, the force exerted on piston B is multiplied in the ratio A_B/A_A. This force multiplication is achieved at the expense of a larger piston stroke at the pedal compared with that of the brake piston.

Number Systems

As well as decimal (or *base-10*) numbers, as a motorsport engineer you will come across hexadecimal (*base-16*) and binary (*base-2*) numbers. You should understand the principles of each system and be able to convert manually between each system and decimal.

Binary

Binary numbers use zeroes and ones to represent a value. Each digit in a binary number is a *bit*. Counting in binary follows the same principle as counting in decimal: the first digit (the one on the right) is incremented by 1 for each subsequent number. When any digit has reached its maximum value, the next increment resets the digit to zero and the digit to the left is incremented by one.

Use this counting method to complete the following table:

Decimal	Binary
0	0
1	1
2	10
3	11
4	100
5	101
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	
20	

Decimal	Binary
21	
22	
23	
24	
25	
26	
27	
28	
29	
30	
31	
32	
33	
34	
35	
36	
37	
38	
39	
40	
41	

Just as each digit in a decimal / base-10 number represents a power of 10, each digit in binary / base-2 represents a power of 2. In the examples below, the subscripts 10 and 2 indicate the base.

Decimal example:

$$3908_{10} = 3 \times 10^3 \ + \ 9 \times 10^2 \ + \ 0 \times 10^1 \ + \ 8 \times 10^0$$

Binary to decimal:

$$1010011_2 = 1 \times 2^6 \ + \ 0 \times 2^5 \ + \ 1 \times 2^4 \ + 0 \times 2^3 \ + \ 0 \times 2^2 \ + \ 1 \times 2^1 \ + \ 1 \times 2^0$$

$$= 1 \times 64 \ + \ 0 \times 32 \ + \ 1 \times 16 \ + \ 0 \times 8 \ + \ 0 \times 4 \ + \ 1 \times 2 \ + \ 1 \times 1$$

$$= 64 + 16 + 2 + 1 = \mathbf{83_{10}}$$

Decimal to binary:

The algorithm for decimal to binary conversion is very simple: divide the decimal by 2 and write the *remainder* as the *least significant bit*. The quotient is then divided again by 2 and the remainder is written as the next least significant bit. Repeat until the quotient is 1. Example of conversion of 251_{10} to binary (R represents the remainder of each calculation):

÷ 2	Integer quotient	Remainder	
251 ÷ 2	125	1	(LSB)
125 ÷ 2	62	1	
62 ÷ 2	31	0	
31 ÷ 2	15	1	
15 ÷ 2	7	1	
7 ÷ 2	3	1	
3 ÷ 2	**1**	1	

So decimal 251 in binary is 1111011.

Hexadecimal

The principle of hexadecimal is the same as binary and decimal, only with a base of 16. In general, the 16 symbols used are 0-9 and A-F:

Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Complete the following table by manual counting:

Decimal	**Hexadecimal**
0	0
1	1
2	2
3	3
4	4
5	5
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	
20	

Decimal	**Hexadecimal**
21	
22	
23	
24	
25	
26	
27	
28	
29	
30	
31	
32	
33	
34	
35	
36	
37	
38	
39	
40	
41	

Hex to decimal conversion follows the same principle as binary to decimal. In hexadecimal each digit represents a power of 16:

$$A3D_{16} = 10 \times 16^2 + \ 3 \times 16^1 + 13 \times 16^0$$
$$= 10 \times 256 + 3 \times 16 + 13 \times 1$$
$$= 2560 + 48 + 13 = 2621_{10}$$

Conversion between hex and binary is simpler than between hex and decimal, because each group of 4 binary digits maps to a single hexadecimal digit.

Find out how to use Excel to convert between decimal and binary and between decimal and hexadecimal.

International Numeric Conventions

You should be aware of the differences in numeric conventions when working in other languages. As well as correctly reading numbers, differences in local number formats can be an important issue for programming and software use.

Number Formats

The table below shows the usual decimal and thousand separators in some relevant locales as well as the ISO standard.

France	4 967 295,000
Germany	4.967.295,000
Italy	4.967.295,000
UK	4,967,295.000
USA	4,967,295.000
ISO 31-0	4 967 295.000 or 4 967 295,000

Time Formats

The 12-hour clock is generally used in English-speaking countries, except when unambiguity and/or precision are of concern. Most of the non-English speaking world use the 24-hour clock, although the 12-hour clock may be used in casual speech in these countries.

Date Formats

The table below shows the usual or standard date format in some relevant locales as well as the ISO standard.

France	dd/mm/yyyy
Germany	yyyy-mm-dd
Italy	dd/mm/yyyy
UK	dd/mm/yy
USA	mm/dd/yy or mm/dd/yyyy
ISO 8601	yyyy-mm-dd

US / Imperial units

The USA is one of only three countries worldwide that have not adopted the metric system as their official system of weights and measures. Since the USA is a significant player in the motorsport and automotive industries and the source of many racing products and engineering information you will inevitably encounter US units of measure in your motorsport engineering career. For those used to the metric system, US units can be a headache to deal with.

The UK, another important player in motorsport, is *officially* metric but actually uses a mixture of metric and imperial units.

If that wasn't bad enough, US units and UK ("imperial") units are often *similar but not the same…*

We'll review some units that you are likely to encounter, some of which (*) you should commit to memory for quick recall when required (in general, it is only necessary to memorise the "natural" or approximate conversions for these key quantities).

Length

Complete the following table and memorise all the "natural" conversions. For the units in this table, US and UK measures are the same.

UK / US unit	SI conversions (exact)	"natural" conversion	
inch (in or ")		2.54cm or 25.4mm (exactly)	*
foot (ft)		0.3m (approx.)	
mile (m or mi)		1.6km (approx.)	*

Volume

Apart from those which are based on cubic length, there are significant differences between US and imperial volume measures. Complete the following table:

US or Imperial unit	SI conversions (exact)	"natural" conversion	
cubic inch (in^3, cu in, etc.)	(refer to the defined length conversions when using cubic lengths)	15 or 16 cm^3 (approx.)	
cubic foot (ft^3, cu ft, etc.)		28 L (approx.)	
US fluid ounce† (fl oz)		30 mL (approx.)	
Imperial fluid oz (fl oz)		30 mL (approx.)	
US liquid gallon (gal)		3.75 L (approx.)	
Imperial gallon (gal)		4.5 L (approx.)	

† N.b. there is another slightly different fl oz used in the USA for food labelling.

Force, Weight and Mass

The words *weight* and *mass* are often used interchangeably in everyday language, but in engineering language we must be sure to use them correctly: weight is a force (vector), mass is a measure of quantity of matter (scalar). In US / Imperial units, an added confusion arises from the fact the same word is used for the unit of both weight and mass. To distinguish between the two, the terms pound-mass and pound-force can be used.

Complete the following table and memorise the "natural conversions.

US / Imperial unit	SI conversions (exact)	"natural" conversion	
pound-mass (lb, lbm or lb_m)		1 kg = 2.2lb	*
pound-force (lbf or lb_f)		4.44 N	*

Torque and Power

Torque is particularly confusing in US units since it is has the unit foot-pound (ft·lb or ft·lb_f), which is the same as the unit of work. For this reason, many refer to the torque unit as *pound-foot* (lb·ft or lb_f·ft). It is a compound unit, so the exact conversion is given by the constituent units of lb_f and ft. Roughly, 1 W ≈ 0.7376 ft·lb_f/sec = 44.25 ft·lb_f/min.

Power in the US / imperial systems of units is measured in ft·lb_f/sec. Since this is also a compound unit, refer to the constituent units for an exact conversion.

The output power of an engine is typically measured in *horsepower* (*mechanical horsepower* or *imperial horsepower*), which is defined as exactly 550 ft·lb_f/s and is roughly 745.7 W*.

There is also a *metric horsepower* (PS) of approximately the same magnitude as the imperial unit. It is the power required to lift 75 kg a height of 1 m in 1 second, so 1 metric horsepower equals 75 kgf-m per second and is approximately 735.5 W* (98.6% of an imperial mechanical horsepower).

Temperature

Temperature in the US is given in Fahrenheit, F, which is a linear scale defined by the melting point of ice at 32 °F and the boiling point of water at 212 °F, both at sea level and standard atmospheric pressure.

Use these two defined points on the Fahrenheit scale to obtain conversion formulas for Celsius to Fahrenheit and Fahrenheit to Celsius and commit these conversions to memory.